

# Propagation of gravitational waves in a universe with slowly-changing equation of state

*Edmund Schluessel*

## Abstract

An exact solution for the expansion of a flat universe with dark energy evolving according to a simple model is explored. The equation for weak primordial gravitational waves propagating in this universe is solved and explored; gravitational waves in a flat cosmology possessing both a “big bang” singularity and a “big rip” singularity can be described with confluent Heun functions. We develop approximation methods for confluent Heun equations in regimes of interest to gravitational wave astronomers and predict the diminution in gravitational wave amplitude in a universe with both a Big Bang and a Big Rip.

## 1 Introduction

### 1.1 Background

As we approach the beginnings of gravitational wave cosmology [10] it becomes necessary to explore the full range of potential models which might affect our initial observations. Riese et al’s [1] observation of acceleration greatly expanded the parameter space of our models for cosmology but the paucity of observational data coupled with the unknown character of the “dark energy” necessary to make observations consistent with general relativity and the cosmological principle have not helped narrow this space much in the past 16 years. As the decay of short-wavelength gravitational waves is dependent on the equation of state of the background through which they travel, the amplitudes of these waves can be used as evidence for different kinds of cosmic evolution. Some exploration has been done in this field numerically [16] but inconvenient mathematics has made analytic work less common. Analytic work in this field is necessary as a rapidly-evolving understanding of observational data requires models which can be generalized and investigated for easily characterizable phenomena.

Throughout this paper we make the following assumptions:

- Classical, unmodified general relativity. We make no comment as to the nature quantum-gravitational or other processes which might generate primordial gravitational waves.

- A cosmologically flat  $K = 0$  universe. While there is no absolute evidence of the large-scale flatness of the universe, flatness is a good approximation for closed models and remains consistent with observations [42].
- A universe which is, on large scales, homogeneous and isotropic.
- The cosmological constant  $\Lambda = 0$ . We will touch on cases where the effective equation of state  $w_{eff} = -1$ , which are more general than the cosmological constant-dominated de Sitter universe; as argued in [4] and in [33],  $w_{eff} = -1$  is a limiting case for a number of cosmologies.

## 1.2 Review: cosmologies with constant equation of state

Consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe described, for energy density  $\epsilon$  and pressure  $p$ , by the Einstein equations<sup>1</sup>

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (1)$$

Define the relationship  $p/\epsilon \equiv w$ , the “equation of state” for a fluid filling the universe. We then have the well-known Einstein equation relating the scale factor of the universe, the function  $a(t)$ , to the equation of state<sup>2</sup>:

$$2\ddot{a}a + (1 + 3w)\dot{a}^2 = 0. \quad (2)$$

In a flat homogeneous universe this relationship is true for any  $w(t)$ . Excluding the trivial solution  $a = \text{constant}$ , we arrive at the general solution

$$a(t) = a_0 e^{\frac{2}{3} \int \frac{1}{f^t(1+w)d\bar{t}+C} dt} \quad (3)$$

where  $a_0$  is a positive number and  $C$  an arbitrary constant. Whether  $w$  is constant or not, flat FLRW universes will have Hubble parameter

$$\frac{\dot{a}}{a} \equiv H = \frac{2}{3} \frac{1}{\int (1+w) dt + C} \quad (4)$$

and deceleration parameter

$$-\frac{\ddot{a}a}{\dot{a}^2} \equiv q = \frac{1 + 3w}{2}. \quad (5)$$

The cases for a constant  $w \equiv w_0 > -1$  are well known and include the ubiquitous “matter-dominated” ( $w_0 = 0$ ) and “radiation-dominated” ( $w_0 = 1/3$ ) cosmologies thought, until the discovery of acceleration, to provide a complete picture

<sup>1</sup> We use the convention  $G = c = 1$  throughout.

<sup>2</sup> In this introductory section, we will use a dot to denote differentiation with respect to time and a prime to denote differentiation with respect to conformal time. In later sections we will use the comma derivative to denote differentiation, so  $f(x^i)_{,x^j} \equiv \partial f / \partial x^j$ .

of the evolution of the post-inflationary universe. The solutions for constant  $w = w_0$  are given by

$$a(t) = \begin{cases} a_0 (t - t_0)^{\frac{2}{3(1+w_0)}} & w_0 > -1 \\ a_0 e^{H_0 t} & w_0 = -1 \end{cases} \quad (6)$$

for some empirical constant Hubble parameter  $H_0$  and where  $t_0$ , another empirical constant, describes the time since a singularity which for  $w_0 > -1$  represents a Big Bang.

The discovery by Riess et al. [1] that  $q < 0$ , in sharp contrast to the prediction of  $q = 1/2$  for a universe with  $w = 0$ , has been confirmed multiple times [17, 18, 19, 20, 21, 23, 24, 25, 27, 28, 29, 30, 31], implying the present day value of  $w < -1/3$ . A partial response to this problem within the framework of FLRW cosmologies comes in the form of the “phantom energy”-dominated universe described in [3], which has scale factor

$$a_{w < -1}(t) = a_0 |t_0 - t|^{\frac{2}{3(1+w_0)}} \quad (7)$$

for  $w_0 < -1$  and where the constant  $t_0$ , rather than denoting a convergent singularity in the past, instead marks a divergent singularity in the future – a “big rip” where the scale factor goes to infinity in finite time.

It is common to simplify the equations that arise in an expanding universe by making a conformal change in variables as per [34]. We define the new time variable  $\eta$  by

$$dt \equiv a d\eta. \quad (8)$$

In conformal time, the equation describing the scale factor reads

$$a(\eta) = \begin{cases} a_0 |\eta - \eta_0|^{\frac{2}{1+3w_0}} & w_0 \neq -1/3 \\ a_0 e^{H_0 \eta} & w_0 = -1/3 \end{cases}. \quad (9)$$

Note that as the two time coordinates are related by

$$a_0 (\eta - \eta_0) = (t - t_0)^{\frac{1+3w_0}{3(1+w_0)}} \quad (10)$$

the cases  $-1/3 > w_0 > -1$  can be deceptive in terms of character of the singularity when analysis is carried out in  $\eta$ -time.

Indications are that the universe’s acceleration is both recent and increasing [35], that is,  $\dot{q} < 0$ , implying the equation of state is evolving with time. In SECTION 2 we will discuss a solution to 2 which obeys this condition and which as a consequence has both “big bang” and “big rip” singularities.

### 1.3 Gravitational waves in universes with constant equation of state

Equation (1) admits a linear-order tensor perturbation  $\delta G_{\mu\nu}(\nu(t))$  such that

$$\ddot{\nu} + 3\frac{\dot{a}}{a}\dot{\nu} + \frac{n^2}{a^2}\nu = 0 \quad (11)$$

[34, 36] where we have removed a gauge term by demanding  $\delta T_{\mu\nu} = 0$ . As  $\nu$  is part of a tensorial solution to the generalized Laplace equation, we regard  $\nu$  as describing a weak gravitational wave in the sense that  $|\nu| \ll 1$  such that  $\nu^2 \approx 0$ . In conformal time, (11) becomes

$$\nu'' + 2\frac{a'}{a}\nu' + n^2\nu = 0. \quad (12)$$

When our equation of state  $w = w_0$  this is expressed as

$$\begin{cases} \nu'' + \frac{4}{1+3w_0}\frac{1}{\eta_*}\nu' + n^2\nu = 0 & w_0 \neq -1/3 \\ \nu'' + 2H_0\nu' + n^2\nu = 0 & w_0 = -1/3 \end{cases} \quad (13)$$

where we define  $\eta_* = (\eta - \eta_0) \text{sgn}(1 + w_0)$  and restrict our analysis only to the period of cosmological expansion following (in the  $w_0 > -1$  case) or leading to (in the  $w_0 < -1$  case) the singularity. In the radiation-dominated  $w_0 = 1/3$  universe, this equation has the well-known solution

$$\nu_{\text{radiation}} = c_1 \frac{\sin(n\eta_*)}{\eta_*} + c_2 \frac{\cos(n\eta_*)}{\eta_*} = c_1 j_0(n\eta_*) + c_2 y_0(n\eta_*) \quad (14)$$

and in a matter-dominated  $w_0 = 0$  universe [38]

$$\begin{aligned} \nu_{\text{matter}} &= \sqrt{\frac{2}{\pi n}} \left[ c_1 \left( \frac{\sin(n\eta_*)}{n\eta_*^3} - \frac{\cos(n\eta_*)}{\eta_*^2} \right) - c_2 \left( \frac{\cos(n\eta_*)}{n\eta_*^3} + \frac{\sin(n\eta_*)}{\eta_*^2} \right) \right] \\ &= \sqrt{\frac{2}{\pi n}} \frac{n}{\eta_*} [c_1 j_1(n\eta_*) + c_2 y_1(n\eta_*)]. \end{aligned} \quad (15)$$

where  $j_\alpha(x)$  and  $y_\alpha(x)$  are spherical Bessel functions of the first and second kind [37]. The application of Bessel functions as a solution to (12) is trivial but seems not to be in wide discussion in the literature. Therefore for completeness and as a reference to what follows we present the solutions here with a few remarks.

For arbitrary constant  $w$ , (12) is solved by

$$\begin{aligned} \nu_w(\eta_*) &= \eta_*^{\frac{3w-3}{2+6w}} \left[ c_1 J_{\frac{1-3w}{1+3w}}(n\eta_*) + c_2 Y_{\frac{1-3w}{1+3w}}(n\eta_*) \right] \\ &= \sqrt{\frac{2n}{\pi}} \eta_*^{\frac{3w-1}{1+3w}} \left[ c_1 j_{\frac{1-3w}{1+3w}}(n\eta_*) + c_2 y_{\frac{1-3w}{1+3w}}(n\eta_*) \right] \end{aligned} \quad (16)$$

except for  $w = -1/3$  when (assuming  $H_0^2 \ll n^2$ )

$$\nu_{w=-1/3}(\eta_*) = e^{-H_0\eta} \left[ c_1 \sin\left(\sqrt{H_0^2 - n^2}\eta_*\right) + c_2 \cos\left(\sqrt{H_0^2 - n^2}\eta_*\right) \right]. \quad (17)$$

In the asymptotic limit of  $n\eta_* \gg 2 \left| (1 - 3w_0) / (1 + 3w_0)^2 \right|$ , (16) and (17) take on the approximate form

$$v_{w_0}(\eta_*) \approx \frac{a_0}{a} \left[ \bar{c}_1 \sin \left( n\eta_* - \frac{\pi}{2} - \frac{\pi}{4} \frac{1 - 3w}{1 + 3w} \right) + \bar{c}_2 \cos \left( n\eta_* - \frac{\pi}{2} - \frac{\pi}{4} \frac{1 - 3w}{1 + 3w} \right) \right], \quad (18)$$

in other words, the gravitational waves decay in amplitude proportional to the scale factor, as broadly predicted in the theorem by Lifshitz [38]. This result shows analytically the outcome predicted numerically by [16], in particular that a gravitational wave will decay less in a phantom energy-driven universe than in a universe expanding through the influence of ordinary matter. The Bessel functions have the property of a regular singularity at  $\eta_* = 0$  and an irregular singular point at  $\eta_* = \infty$ .

By re-stating (18) in  $t$ -time, it is also obvious gravitational waves in a universe with constant equation of state undergo the same redshift as electromagnetic waves. Noting frequency  $f(t) \propto a^{-1}$ ,

$$\frac{f_{\text{emission}}}{f_{\text{observation}}} = \frac{a_{\text{observation}}}{a_{\text{emission}}}. \quad (19)$$

The amplitude  $A$  of the waves meanwhile diminishes in proportion to the scale factor,

$$\frac{A_{\text{observation}}}{A_{\text{emission}}} = \frac{a_{\text{observation}}}{a_{\text{emission}}}. \quad (20)$$

## 2 Evolution of a universe with simply-changing equation of state

### 2.1 Cosmologies for a linear-function equation of state

#### 2.1.1 Solutions in $t$ -time

Cosmological evolution for a flat universe with constant  $w$  is well-known. We will consider the evolution of a universe with a slightly more complicated equation of state:

$$w = w_0 - w_1 t; \quad (21)$$

and in conformal time, the related universe with

$$w = w_0 - v_1 \eta. \quad (22)$$

where  $w_1$  and  $v_1$  are arbitrary real constants (we do *not* assume that  $w_1$  and  $v_1$  are necessarily small). The solutions for cosmologies with the above equations of state have been explored by Babichev *et al* [4] in the context of purely dark energy-dominated cosmologies; we replicate their work in a more practical formalism in this section and apply it in SUBSECTION 2.2 and SECTION 3.

Substituting (21) into (3) and solving gives us the generic solution

$$a(t) = \left[ e^{\int \frac{1}{-\frac{1}{2}w_1 t^2 + (1+w_0)t + C} dt} \right]^{2/3} \quad (23)$$

but this must be analyzed in cases to get useful results. These cases are distinguished by the discriminant  $\Delta \equiv (1 + w_0)^2 + 2w_1 C$  of the integral  $\int w dt$ :

$$a = \begin{cases} a_0 e^{\frac{4}{3\sqrt{-\Delta}} \tan^{-1} \left( \frac{-w_1 t + (1+w_0)}{\sqrt{2w_1 C - (1+w_0)^2}} \right)} & \Delta < 0 \\ a_0 e^{-\frac{4}{3} \frac{1}{-w_1 t + (1+w_0)}} & \Delta = 0 \\ a_0 \left( -\frac{2}{w_1} \frac{-w_1 t + (1+w_0) - \sqrt{\Delta}}{-w_1 t + (1+w_0) + \sqrt{\Delta}} \right)^{\frac{2}{3}} \frac{1}{\sqrt{(1+w_0)^2 - 2w_1 C}} & \Delta > 0. \end{cases} \quad (24)$$

Each of these, for appropriate choices of  $w_0$ ,  $w_1$  and  $C$ , produces a plausible universe, that is, one with an epoch where  $H$  is positive.

Cosmologies of the first kind in (24) have no singularities in their evolution, but evolve from an initial finite non-zero scale factor in the infinite past to another in the infinite future, the two scale factors separated by a multiplicative factor of  $\exp \left[ 4\pi / \left( 3\sqrt{-2w_1 C - (1 + w_0)^2} \right) \right]$ . The parameter  $C$  cannot be set to zero in such a universe and controls the ratio between the maximum and minimum scale factor. Cosmologies of this kind model an expanding universe only when  $w_1 > 0$ .

Cosmologies of the second kind in (24) have the tuned value  $C = (1 + w_0)^2 / 2w_1$ . This solution to (2), when and only when  $w_1 < 0$ , has two regions which could model an expanding universe. One expands from an initial finite non-zero scale factor in the infinite past to a big rip at  $t = (1 + w_0) / w_1$ ; the other expands asymptotically from a big bang at  $t = (1 + w_0) / w_1$  to a finite scale factor in the infinite future.

A cosmology of the third kind in (24) can evolve in a manner most like that of our own universe. In this model the parameter  $C$  can be set to zero without loss of generality, giving a simplified expression of

$$a = a_0 \left( \frac{2t}{2(1 + w_0) - w_1 t} \right)^{\frac{2}{3} \frac{1}{1+w_0}}. \quad (25)$$

When  $w_1 > 0$  this cosmology has a region marked by an initial convergent “big bang” singularity, a period of expansion, and a divergent “big rip” singularity; where  $w_0 > -1$  the “big bang” takes place at  $t = 0$ . When  $w_1 < 0$  this cosmology contains two different regions of expansion similar to that of the cosmology of the second kind.

### 2.1.2 Related models in conformal time

While the relationship between coordinates  $t$  and  $\eta$  is generally not simple, we can work out solutions to (2) analogous to (24) by use of the model equation of state (22). These solutions are simply, where  $\bar{\Delta} \equiv (1 + 3w_0)^2 + 6v_1 \bar{C}$

$$a(\eta) = \begin{cases} a_0 e^{\frac{4}{\sqrt{-\bar{\Delta}}} \tan^{-1}\left(\frac{1+3w_0-3v_1\eta}{\sqrt{-\bar{\Delta}}}\right)} & \bar{\Delta} < 0 \\ a_0 e^{\frac{4}{3v_1\eta-(1+3w_0)}} & \bar{\Delta} = 0 \\ a_0 \left(\frac{2\eta}{2(1+3w_0)-3v_1\eta}\right)^{\frac{2}{1+3w_0}} & \bar{\Delta} > 0 \end{cases} \quad (26)$$

where in the third case we have set  $\bar{C} = 0$  analogously to case (25) above.

## 2.2 Can the double-singularity model describe our universe?

While the cosmology of the third kind in (24) is qualitatively desirable to model a universe with slowly-evolving dark energy, it should also be quantitatively compatible with observations. For an equation of state (21) and setting  $w_0 = 0$ , the system created by (4) and (5) is solved by

$$w_1 = -\frac{H_0}{6} (2q_0 - 1) \left(q_0 + \frac{5}{2}\right) \quad (27)$$

$$t_{\text{now}} = \frac{2}{H_0 \left(q_0 + \frac{5}{2}\right)}. \quad (28)$$

For the current best values for the Hubble parameter  $H_0 = 67.80 \pm 0.77 \text{ km/s} \times \text{Mpc}$  [5] and  $q_0 = -0.53^{+0.17}_{-0.13}$  [6] we obtain

$$\begin{aligned} w_1 &= 1.5^{+0.1}_{-0.0} \times 10^{-18} \text{ s}^{-1} \\ t_{\text{now}} &= 4.6 \pm 0.4 \times 10^{17} \text{ seconds} = 15 \pm 1 \text{ Gya} \\ t_{\text{rip}} &= 1.3^{+0.1}_{-0.0} \times 10^{18} \text{ seconds} = 41 \pm 3 \text{ Gya} \end{aligned} \quad (29)$$

which is easily compatible with the current observations for the age of the universe [5] and indicates roughly 37% of the universe's lifespan has passed before a “big rip” 26 billion years in the future.

The  $\eta$ -time model is more limited than our  $t$ -time model: the central, “bang-rip” region of the solution can only explain  $-1/2 < q < 1/2$  if we assume  $w_0 = 0$ . Nonetheless it will serve our purpose in beginning the exploration of gravitational waves in such cosmologies.

## 3 Gravitational waves in a universe with simply-changing equation of state

### 3.1 Evolution of gravitational waves in $\eta$ -time

The origin of the universe, whatever form it took, likely left an imprint in the form of a gravitational wave background generated by the universe's primordial physical processes. The upcoming Next Gravitational-wave Observatory (NGO, formerly LISA) should be able to detect high-frequency relic gravitational waves

[7]. We have seen in SUBSECTION 1.3 how a universe with constant equation of state preserves the spectrum of relic gravitational waves over time. The same will be true in a universe with an evolving equation of state.

Plugging the third case of (26) into (11) gives us

$$\nu_{,\eta,\eta} + \frac{4}{1+3w_0} \frac{1}{\eta \left(1 - \frac{3v_1}{2(1+3w_0)}\eta\right)} \nu_{,\eta} + n^2 \nu = 0. \quad (30)$$

With a changes of variables

$$\xi \equiv \frac{3v_1}{2(1+3w_0)}\eta \quad (31)$$

we obtain

$$\xi(\xi-1)\nu_{,\xi,\xi} - 2(1+\beta)\nu_{,\xi} + \omega^2\xi(\xi-1)\nu = 0 \quad (32)$$

where

$$\beta \equiv \frac{1-3w_0}{1+3w_0} \quad (33)$$

$$\omega \equiv 2(1+3w_0)n/3v_1. \quad (34)$$

We recognize (32) as being in the form of the Generalized Spherical Wave Equation (GSWE) [9]. If we go on to change the independent variable

$$\mu = e^{-i\omega\xi}\nu \quad (35)$$

we obtain, where  $i$  is the imaginary unit,

$$\mu_{,\xi,\xi} + \left(2i\omega + \frac{2(1+\beta)}{\xi} - \frac{2(1+\beta)}{\xi-1}\right)\mu_{,\xi} - \frac{2i(1+\beta)\omega}{\xi(\xi-1)}\mu = 0 \quad (36)$$

which we recognize as the confluent Heun equation [8].

Formally, we can say that (36) has particular solution

$$\mu_1 = \text{Hc} \left( \begin{array}{ccc} \frac{1}{2}i\omega, & 2(1+\beta), & -2(1+\beta) \\ 0, & 2i(1+\beta)\omega & \end{array} ; \xi \right) \quad (37)$$

and therefore (30) has general solution

$$\nu(\eta) = \left\{ \begin{array}{l} e^{i\eta\eta} \text{Hc} \left( \begin{array}{ccc} \frac{1}{2}i\omega, & 2(1+\beta), & -2(1+\beta) \\ 0, & 2i(1+\beta)\omega & \end{array} ; \frac{3v_1}{2(1+3w_0)}\eta \right) \times \\ \times \left[ C_1 + C_2 \int^\eta \frac{e^{-2in\chi} a^2(\chi)}{\left[ \text{Hc} \left( \begin{array}{ccc} \frac{1}{2}i\omega, & 2(1+\beta), & -2(1+\beta) \\ 0, & 2i(1+\beta)\omega & \end{array} ; \frac{3v_1}{2(1+3w_0)}\chi \right) \right]^2} d\chi \right] \end{array} \right\} \quad (38)$$

where Hc is the confluent Heun function  $\text{Hc}(p, \gamma, \delta, \alpha, \sigma; z)$  as defined in [8]. This solution is not especially useful in and of itself, as (37) cannot generally



be expressed as a finite series of simpler functions [8, 9] although much work has been put into infinite-series expressions for Heun functions (see for example [11, 12, 13]). Moreover as the most physically interesting cases of  $\beta$  are integer values  $\beta = \{0, 1\}$  corresponding to  $w_0 = \{1/3, 0\}$ , we need to use the Frobenius method to express the second solution.

Leaver [9] provides the oldest series solutions for  $\mu$  about the points  $\xi = 0$  and  $\xi = 1$ , those of Baber and Hassé. Equation (36) has particular three-term recurrent series solution

$$\mu_1 = \sum_{m=0}^{\infty} b_m \xi^m \quad (39)$$

$$b_1 + \omega b_0 = 0 \quad (40)$$

$$\left\{ \begin{array}{l} - (m+2)(m+3+2\beta)b_{m+2} + \\ + [m^2 + (1-2\omega)m - 2(2+\beta)\omega]b_{m+1} + \\ + 2\omega m b_m \end{array} \right\} = 0 \quad (41)$$

about  $\xi = 0$ . The “decaying” mode of the gravitational waves in our model universe diverges as  $\eta^{-1-2\beta}$ . In contrast to the solutions for a universe with constant equation of state, the second solution includes a  $(\ln \eta)$ -term when  $2\beta$  is an integer; this solution can be obtained by the Frobenius method.

While in general (32) is difficult to work with analytically, in the particular case of high-frequency gravitational waves, that is  $\omega \gg 1$ , we can say to good approximation

$$\nu \approx (\xi - 1)^{1+\beta} (\omega \xi)^{-\beta} [C_1 j_\beta(\omega \xi) + C_2 y_\beta(\omega \xi)]; \quad (42)$$

for some constants  $C_1, C_2$ ; this approximation is fully derived in the APPENDIX. This approximation (42) is nearly identical in form to (16); indeed, recalling  $\omega \xi = n\eta$  we can go on to approximate

$$\nu \approx a^{-1} \left[ C_1 \sin \left( n\eta - \frac{\pi}{2} (1 + \beta) \right) + C_2 \cos \left( n\eta - \frac{\pi}{2} (1 + \beta) \right) \right]. \quad (43)$$

The solution for  $\nu$ , at least in approximation, goes to zero in finite time. This implies that gravitational waves in a universe expanding with scale factor as in the third case of (26) must always decay faster than those with identical initial conditions in a universe with analogous constant equation of state. Because the time parameter  $\eta$  is defined by this scale factor function, though, we cannot easily compare the functions directly and need to return to  $t$ -time in order to best discuss how our dark energy model would affect gravitational wave observations.

### 3.2 Evolution of gravitational waves in $t$ -time

In universes with a single singularity as discussed in SUBSECTION 1.2, relating solutions to the wave equation (11) to corresponding solutions in conformal time is trivial, making solving the equations in  $\eta$  mathematically desirable. In the

cosmologies under our consideration, the relationship between  $t$  and  $\eta$  is more complicated, requiring the use of inverse functions not expressible in terms of standard functions and producing no mathematical advantages in the solution of the associated differential equations. Therefore we briefly explore solutions to (11) in  $t$ -time for our equation of state (21).

In  $t$ -time, our gravitational wave equation for a universe with equation of state (21) reads

$$\ddot{\nu} + \frac{4}{t[2(1+w_0) - w_1 t]} \dot{\nu} + \frac{n^2}{a_0^2} \left( \frac{2t}{2(1+w_0) - w_1 t} \right)^{-\frac{4}{3} \frac{1}{1+w_0}} \nu = 0. \quad (44)$$

Introducing the following notation to simplify our expressions:

$$x \equiv \frac{w_1}{2(1+w_0)} t \quad (45)$$

$$v \equiv (1+w_0) \frac{n}{a_0} \left( \frac{w_1}{2} \right)^{\frac{3+2\beta}{2+\beta}} \quad (46)$$

(the scale factor takes on the simple form

$$a = a_0 \left( \frac{2}{w_1} \frac{x}{1-x} \right)^{\frac{1+\beta}{2+\beta}} \quad (47)$$

in this notation) we arrive at

$$x(x-1) \nu_{,x,x} - 3 \frac{1+\beta}{2+\beta} \nu_{,x} - v^2 (1-x)^{\frac{4+3\beta}{2+\beta}} x^{-\frac{\beta}{2+\beta}} \nu = 0. \quad (48)$$

In the case of  $w_0 = 1/3 \iff \beta = 0$ , this gives us another GSWE:

$$x(x-1) \nu_{,x,x} - \frac{3}{2} \nu_{,x} - v^2 (1-x)^2 \nu = 0 \quad (49)$$

with corresponding solution

$$\nu = e^{vx} \left[ C_1 \text{Hc} \left( \begin{matrix} v/2, & 3/2, & -3/2, \\ v/2, & 3v/2 + v^2, & \end{matrix} ; x \right) + C_2 (x-1)^{5/2} \text{Hc} \left( \begin{matrix} v/2, & 3/2, & 7/2, \\ 5/2 + v/2, & v^2 + 3v/2 + 15/4, & \end{matrix} ; x \right) \right] \quad (50)$$

(note the solution is separated into a confluent Heun function with purely real parameters and an exponential function with purely real power, in contrast to the analogous  $\eta$ -time case). This solution can be approximated, using the same method as for (42), as

$$\nu \approx \left( \frac{1-x}{x} \right)^{\frac{1}{2}} \left[ c_1 \cos \left( \frac{1}{4} \pi - \frac{1}{2} \pi v + v \sqrt{x(1-x)} + v \sin^{-1} \sqrt{x} - \frac{1}{v} \frac{\sqrt{x}(x-6)}{48(1-x)^{3/2}} \right) + c_2 \cos \left( \frac{1}{4} \pi - \frac{1}{2} \pi v - v \sqrt{x(1-x)} - v \sin^{-1} \sqrt{x} + \frac{1}{v} \frac{\sqrt{x}(x-6)}{48(1-x)^{3/2}} \right) \right] \quad (51)$$

Also of note is the phantom-energy case  $w_0 = -7/3 \iff \beta = -4/3$ , which gives GSWE

$$x(x-1)\nu_{,x,x} + \frac{3}{2}\nu_{,x} - v^2x^2\nu = 0 \quad (52)$$

and therefore has exact solution

$$\nu = e^{vx} \left[ C_1 \text{Hc} \left( \begin{matrix} v/2, & -3/2, & 3/2, & ; x \end{matrix} \right) + C_2 (x-1)^{-1/2} \text{Hc} \left( \begin{matrix} v/2, & -3/2, & 1/2, & ; x \end{matrix} \right) \right]. \quad (53)$$

The case of greatest physical interest,  $w_0 = 0 \iff \beta = 1$ , does not give us an equation apparently solvable in terms of Heun functions but we can write the approximate solution

$$\nu \approx \left( \frac{1-x}{x} \right)^{\frac{1}{2} \frac{1+2\beta}{2+\beta}} \left[ C_1 J_{\beta+\frac{1}{2}} \left( (2+\beta) v \left( \frac{x}{1-x} \right)^{\frac{1}{2+\beta}} \right) + C_2 Y_{\beta+\frac{1}{2}} \left( (2+\beta) v \left( \frac{x}{1-x} \right)^{\frac{1}{2+\beta}} \right) \right] \quad (54)$$

which can be further approximated as

$$\nu \approx a^{-1} \left[ c_1 \cos \left( (2+\beta) v \left( \frac{x}{1-x} \right)^{\frac{1}{2+\beta}} + \frac{\pi}{2} \beta \right) + c_2 \sin \left( (2+\beta) v \left( \frac{x}{1-x} \right)^{\frac{1}{2+\beta}} + \frac{\pi}{2} \beta \right) \right] \quad (55)$$

and in the case of  $w_0 = 0 \iff \beta = 1$  we have

$$\nu \approx \left( \frac{1-x}{x} \right)^{1/2} \left[ C_1 J_{3/2} \left( 3v \left( \frac{x}{1-x} \right)^{1/3} \right) + C_2 Y_{3/2} \left( 3v \left( \frac{x}{1-x} \right)^{1/3} \right) \right]. \quad (56)$$

This approximation suggests the frequency of gravitational waves will decrease with time until  $x = 2/3$  after which time the frequency diverges to infinity as the amplitude converges to zero. This represents an artefact of the approximation scheme rather than a gravitational “ultraviolet catastrophe”, as will be seen in SUBSECTION 3.4.

### 3.3 Amplitude of gravitational waves

From (29) and (54-56) we are prepared to make quantitative predictions about the evolution of gravitational waves in our accelerating universe. Let  $A = A(\nu(t^*))$  denote the RMS amplitude of a gravitational wave described by  $\nu(t)$  over one cycle evaluated in an interval about some time  $t^*$ ; let  $F = F(\nu(t^*))$  denote the reciprocal of the period of a cycle of the wave evaluated on this same interval. Let symbols with a bar denote evaluations made in an accelerating universe modeled by (25) and let symbols with no bar denote evaluations made

in a corresponding Friedmann universe with identical parameters except  $w_1 = 0$ . Then the following relation describes the decay of gravitational waves in our universe versus that in a classical Friedmann universe:

$$\frac{\bar{A}_{\text{observed}}/\bar{A}_{\text{emitted}}}{A_{\text{observed}}/A_{\text{emitted}}} = \left( \frac{1 - x_{\text{observed}}}{1 - x_{\text{emitted}}} \right)^{\frac{1+\beta}{2+\beta}} \quad (57)$$

$$= \left[ \frac{2(1 + w_0) - w_1 t_{\text{observed}}}{2(1 + w_0) - w_1 t_{\text{emitted}}} \right]^{\frac{2}{3} \frac{1}{1+w_0}}. \quad (58)$$

By (29) we live at  $x \approx .37$ , so a primordial gravitational wave will have a decayed to a strength, relative to that in the analogous FLRW case, of

$$\bar{A}/A \approx (1 - w_1 t/2)^{2/3} \sim .73, \quad (59)$$

in other words, a gravitational wave created at  $t_{\text{emitted}} \approx 0$  and evolving in our accelerating universe will have only 73% the amplitude today it would have in a universe evolving without dark energy.

### 3.4 Gravitational waves at the “big rip” singularity

As we inhabit a universe closer to its beginning than its end, for purposes of astronomy we were, in the previous subsection, primarily interested in approximations for approximations about  $x = 0$ . In terms of mathematical techniques, though, the two regular singularities of a Heun function at  $x = 0$  and  $x = 1$  are identical and therefore we can just as easily discuss the fate of gravitational waves as the scale factor diverges to infinity.

For sake of simplicity consider the case  $w_0 = 1/3 \iff \beta = 1$ . Applying the same techniques as brought us from (49) to (51) approximating about  $x = 1$  we have

$$\nu \approx e^{v(1-x)} x^{3/4} \left\{ \begin{array}{l} C_1 \left[ {}_1F_1 \left( -\frac{v}{2} - \frac{3}{4}; -\frac{3}{2}; 2v(x-1) \right) \right] + \\ + C_2 \psi \left( -\frac{v}{2} - \frac{3}{4}; -\frac{3}{2}; 2v(x-1) \right) \end{array} \right\}. \quad (60)$$

where  ${}_1F_1(a, b, w)$  is Kummer’s degenerate hypergeometric function and  $\psi(a, b, w)$  is Tricomi’s function [37, chapter 13]. Both branches of this function are continuous in their derivatives at  $x = 1$ . This result is at first glance surprising, as it implies that gravitational waves at the Big Rip diminish to zero amplitude and zero frequency – but then resurge, increasing in frequency and amplitude and carrying information across the singularity and into whatever comes afterward!

Such a result is not so alarming in the wider context of gravitational physics however. The extensibility of a metric across a regular singularity is the motivation behind the well-known Kruskal representation of the Schwarzschild solution describing a black hole [2]. The analogy between the Big Rip and black hole physics is qualitatively obvious – what, after all, is the Big Rip other than an infinite-redshift surface?

## 4 Conclusions

Our statement of the scale factor of a universe driven by slowly-evolving dark energy is mathematically more convenient than the form given in [4]. The two-singularity Big Bang-Big Rip model is more realistic than the Big Rip cosmology in its purest,  $w = \text{constant}$  form. The current data also suggests the Bang-Rip cosmology is equally plausible as one with only a Big Rip.

It is frequently the case that developments in physics and mathematics run neck-and-neck. The beginning of the recent more widespread exploration of Heun functions (1995, with [39]) coincides more or less with the discovery of cosmic acceleration & dark energy (1998, with [1]) so from a philosophical standpoint our result in SECTION 3 is not surprising. Greater awareness of this class of functions will lead to greater exploration within the community of mathematical physicists of systems with multiple singularities.

While the diminution in gravitational wave amplitude we predict in SUBSECTION 3.2 is not great, it should be detectable by planned gravitational wave observatories. As predicted by [10], gravitational waves can be used as standard sirens for measuring the cosmological equation of state as long as they can be associated with electromagnetic components.

The interpretation of the Big Rip as a one-way membrane analogous to the infinite-redshift surface of a black hole is, as far as we are aware, novel to this work. The continuity of the gravitational wave function across the singularity might loosely be interpreted as transmitting information to another universe which begins with a Big Bang at  $t = t_{rip}$ , but the fact of the continuity of the metric means we must be more precise about the meaning of “universe”. Without doubt, the Big Rip remains a catastrophic event for matter, although transmission of gravitational waves generated by the motion of that matter across the singularity might create a kind of immortality in the form of persistent perturbations to the  $t > t_{rip}$  region of spacetime.

It is likely no coincidence that the main area where confluent Heun functions have appeared as solutions in gravitational physics is in the perturbations of the Kerr black hole [15]. An interesting project for further mathematical exploration might be to see if a coordinate transformation will relate the Big Bang-Big Rip cosmology to the Kerr metric; if so, cosmic acceleration might be described as analogous to a cosmic angular momentum in a sense very different from that of the “rotating universe” Bianchi VII models.

## Appendix: approximating confluent Heun functions in the high-frequency regime

The general theory of Heun functions remains largely incomplete [39] and even the computational tools to investigate them numerically are still being developed [13]. Therefore for analytic work it is necessary to make use of special values of the Heun functions’ parameters in order to relate the results to better-known cases.

To derive the approximations we make use of in this work, we are inspired by the direction of [40], who make use of transformations of the confluent Heun equation in order to separate out one of the equation's singular points. Consider the general form of the confluent Heun equation:

$$f_{,z,z} + \left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right) f_{,z} + \frac{4p\alpha z - \sigma}{z(z-1)} f = 0. \quad (61)$$

If we have  $\gamma = \sigma = 0$ , or  $\delta = 4p\alpha - \sigma = 0$ , then the singularity at  $z = 0$  or  $z = 1$  respectively vanishes and (61) is reduced to a confluent hypergeometric equation. Our equation (36) and the confluent Heun equation related to (49) cannot be transformed by form-preserving transformations [41] into simpler equations, but they can be transformed into forms which are *almost* reducible, and which can be approximately solved.

Let  $g \equiv (z-1)^{-\delta/2} f$ . Then we have

$$g_{,z,z} + \left(4p + \frac{\gamma}{z}\right) g_{,z} + \frac{1}{z(z-1)^2} \left[ \begin{aligned} &4p(\alpha - \delta/2)(z-1)^2 + \\ &+ (4p\alpha - 2p\delta - \sigma - \gamma\delta/2 + \delta/2 - \delta^2/4)(z-1) + \\ &+ \delta/2 - \delta^2/4 \end{aligned} \right] g = 0. \quad (62)$$

In the case of (36) for  $w_0 = 1/3$  the fact that  $\alpha = 0, \gamma = -\delta = 2, \sigma = 2\gamma p$  means (62) reduces to

$$g_{,z,z} + \left(4p + \frac{2}{z}\right) g_{,z} + \frac{4p(z-1)^2 - 2}{z(z-1)^2} g = 0 \quad (63)$$

where  $p$  is an arbitrary imaginary number. In any practical case for gravitational wave astronomy, the quantity  $\|4p(z-1)^2\| \gg 2$  (recall  $\omega\xi = n\eta$ ) so we can say

$$zg_{,z,z} + (4pz + 2)g_{,z} + 4pg \approx 0. \quad (64)$$

Further simplifying matters, the solution to (64) is of the form  $g \approx C_1 ({}_1F_1(-a, -2a, w)) + C_2 \psi(-a, -2a, w)$  which is a well-known identity for the Bessel functions, where  ${}_1F_1(a, b, w)$  is Kummer's degenerate hypergeometric function and  $\psi(a, b, w)$  is Tricomi's function [37, chapter 13] and  $a$  and  $b$  are arbitrary parameters.

The necessary condition for the approximation we have just made use of is that the  $p$ -dependent terms cancel out everywhere except the leading term in the polynomial forming the numerator of the third term in (62); that is, assuming the parameters  $\alpha, \gamma$  and  $\delta$  are independent of  $p$ , we need  $4p\alpha - 2p\delta - \sigma = 0$  (in terms of Leaver's parameters for the GSWE as used in [9] this is simply the condition  $B_3 = 0$ ). If we impose this condition then (62) leads to the equation

$$zg_{,z,z} + (4pz + \gamma)g_{,z} + 4p\left(\alpha - \frac{\delta}{2}\right)g \approx 0 \quad (65)$$

which has solution

$$g \approx C_1 \left( {}_1F_1\left(\alpha - \frac{\delta}{2}; \gamma; -4pz\right) \right) + C_2 \psi\left(\alpha - \frac{\delta}{2}; \gamma; -4pz\right). \quad (66)$$

For (36) we have  $\alpha = 0, \gamma = -\delta, p = i\omega/2, z = \xi$  which leads directly to the Bessel function solution (42).

Recalling the asymptotic form for the Bessel functions for large  $w$  [37, chapter 9]

$$C_1 J_\kappa(w) + C_2 Y_\kappa(w) \approx \sqrt{\frac{\pi}{2w}} \left[ C_1 \sin\left(w - \frac{\pi}{4} - \frac{\pi}{2}\kappa\right) + C_2 \cos\left(w - \frac{\pi}{4} - \frac{\pi}{2}\kappa\right) \right] \quad (67)$$

lets us then obtain (43).

In the case of (49) the approximation follows mostly the same pattern as above. Applying the same transformation to the confluent Heun equation and approximating in the limit of large  $v$  we arrive at the confluent hypergeometric equation

$$xg_{,x,x} + \left(2vx + \frac{3}{2}\right)g_{,x} + \left(v^2 + \frac{3}{2}v\right)g \approx 0. \quad (68)$$

This equation can be solved in terms of the parabolic cylinder function  $U(a, z)$  as defined in [37, chp 19] and approximated, using Darwin's approximation, as

$$g \approx e^{-vx} \frac{(1-x)^{-\frac{1}{4}}}{\sqrt{-4vx}} \left[ c_1 \cos\left(\frac{1}{4}\pi - \frac{1}{2}\pi v + v\sqrt{x(1-x)} + v\sin^{-1}\sqrt{x}\right) + c_2 \cos\left(\frac{1}{4}\pi - \frac{1}{2}\pi v - v\sqrt{x(1-x)} - v\sin^{-1}\sqrt{x}\right) \right]; \quad (69)$$

thus we arrive at (51).

Equation (48) is more general than the GSWE and so solutions cannot necessarily be expressed as Heun functions, but if we assume  $\beta$  is a rational number such that  $(1 + \beta)/(2 + \beta) = u/v$  and  $u, v \in \mathbb{Z} \setminus 0$  we can transform it into an equation with polynomial coefficients solvable by the Frobenius method with the change of variables  $x \rightarrow \tau^v/(\tau^v + 1)$ ; this maps the interval  $x \subseteq (0, 1)$  to the interval  $\tau \subseteq (0, \infty)$ . While this transformation introduces  $v - 1$  new singularities at  $\tau = \sqrt[v]{-1}$  these are all at non-real values of  $\tau$  and, as the form of the transformed equation shows, all interchangeable through Möbius transformations:

$$\tau(\tau^v + 1)^4 \nu_{,\tau,\tau} + (\tau^v + 1)^3 [(1 + v + 3u)\tau^v + (1 - v + 3u)] \nu_{,\tau} + v^2 \tau^{2v-2u-1} \nu = 0. \quad (70)$$

This general form, while well-suited to numerical work, does not produce intuitive analytic results so we turn again to approximation. If we assume  $0 \leq \beta \leq 1$ , as we must for physically-interesting cases, then to order  $\tau^{v-1}$  we have a wave equation of the same form as (12), leading to the approximate solution (54).

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